Int. J. Solids Structures Vol. 17, pp. 541–542, 1981
Printed in Great Britain.

0020-7683/81/050541–02$02.00/0

NOTE ON GREEN'S FUNCTION FOR A SEMICIRCULAR PLATE

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(*Received* 27 *May* 1980; *in revised form* 11 *August* 1980)

Abstract—A. K. Naghdi's [1] closed-form Green's function for a semicircular plate, clamped around the curved edge and simply supported along its diameter, is re-derived by the method of images.

Consider a semicircular plate of radius R which is simply supported along its diameter and clamped around its curved edge. In terms of dimensionless polar coordinates $\rho = r/R$ and θ, the plate is described by $0 \le \rho \le 1$, $0 \le \theta \le \pi$. The plate is acted upon by a transverse concentrated force P at the point (ρ_0, θ_0), where $0 < \rho_0 < 1, 0 < \theta_0 < \pi$. The resulting transverse displacement w of the plate is given by

$$w = \frac{PR^2}{D} G(\rho, \theta; \rho_0, \theta_0) \tag{1}$$

where D is the flexural rigidity of the plate and G is the biharmonic Green's function for the semicircular plate. The latter function is to be determined as a solution of the differential equation

$$\left.\begin{aligned} &\nabla^2\nabla^2 G = \frac{\delta(\rho-\rho_0)}{\rho_0}\delta(\theta-\theta_0),\ 0<\rho<1,\ 0<\theta<\pi, \\ &\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}, \end{aligned}\right\} \tag{2}$$

subject to the boundary conditions

$$G = 0,\ \frac{\partial G}{\partial \rho} = 0, \quad \text{at } \rho = 1,\ 0 \le \theta \le \pi; \tag{3}$$

$$G = 0,\ \nu\frac{\partial^2 G}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial G}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial\theta^2} = 0 \text{ at } 0 \le \rho \le 1,\ \theta = 0,\ \theta = \pi. \tag{4}$$

In (4), ν denotes Poisson's ratio. A closed-form result for the Green's function G was recently obtained by Naghdi ([1], eqn 33). In this note Naghdi's result is re-derived in a simpler manner using the method of images.

In our approach the Green's function G is continued to the full circular region $0 \le \rho \le 1$, $-\pi \le \theta \le \pi$, by defining

$$G(\rho, \theta; \rho_0, \theta_0) = -G(\rho, -\theta; \rho_0, \theta_0), \quad 0 \le \rho \le 1,\ -\pi \le \theta \le 0. \tag{5}$$

Then the continued function G will satisfy the differential equation

$$\nabla^2\nabla^2 G = \frac{\delta(\rho-\rho_0)}{\rho_0}\delta(\theta-\theta_0) - \frac{\delta(\rho-\rho_0)}{\rho_0}\delta(\theta+\theta_0),\quad 0 \le \rho < 1,\ -\pi \le \theta \le \pi \tag{6}$$

and the boundary conditions

$$G = 0,\ \frac{\partial G}{\partial\rho} = 0 \text{ at } \rho = 1,\ -\pi \le \theta \le \pi. \tag{7}$$

For reasons of symmetry the solution of the problem (6), (7) will automatically satisfy the boundary conditions (4). Clearly, the function G represents the deflection of a clamped circular plate due to opposite concentrated loads at the points $(\rho_0, \pm\theta_0)$. Hence, G can be expressed in terms of the Green's function G_0 for a circular plate clamped around its edge, viz.

$$G(\rho, \theta; \rho_0, \theta_0) = G_0(\rho,\theta;\rho_0,\theta_0) - G_0(\rho,\theta;\rho_0, -\theta_0). \tag{8}$$

The Green's function G_0 is well known from early investigations by Michell[2] and Melan[3],

$$G_0(\rho,\theta;\rho_0,\theta_0) = \frac{1}{8\pi}(\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\theta-\theta_0)) \log\left[\frac{\rho^2+\rho_0^2-2\rho\rho_0\cos(\theta-\theta_0)}{1+\rho^2\rho_0^2-2\rho\rho_0\cos(\theta-\theta_0)}\right]^{1/2}$$
$$+\frac{1}{16\pi}(1-\rho^2)(1-\rho_0^2). \tag{9}$$

Thus we obtain as our final result

$$G(\rho,\theta;\rho_0,\theta_0) = \frac{1}{8\pi}(\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\theta-\theta_0)) \log\left[\frac{\rho^2+\rho_0^2-2\rho\rho_0\cos(\theta-\theta_0)}{1+\rho^2\rho_0^2-2\rho\rho_0\cos(\theta-\theta_0)}\right]^{1/2}$$
$$-\frac{1}{8\pi}(\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\theta+\theta_0)) \log\left[\frac{\rho^2+\rho_0^2-2\rho\rho_0\cos(\theta+\theta_0)}{1+\rho^2\rho_0^2-2\rho\rho_0\cos(\theta+\theta_0)}\right]^{1/2}. \tag{10}$$

It is readily seen that $G = 0$ and $\partial^2 G/\partial\theta^2 = 0$ at $0 \le \rho \le 1$, $\theta = 0$, $\theta = \pi$; hence, our solution does satisfy the boundary conditions (4) as already predicted above. The present closed-form result for the Green's function G is in accordance with ([1], eqn 33) after some re-arrangement of terms.

Finally, it is pointed out that the method of images can also be used to construct the Green's function G_n for a plate sector of angle π/n, $n = 1, 2, 3, \ldots$, clamped around its curved edge and simply supported along its boundaries $\theta = 0$ and $\theta = \pi/n$. For example, in the cases $n = 2$ and $n = 3$ corresponding to a quarter sector and a 60°-sector, respectively, it is easily found that the Green's functions G_2 and G_3 are given by

$$G_2(\rho,\theta;\rho_0,\theta_0) = G_0(\rho,\theta;\rho_0,\theta_0) - G_0(\rho,\theta;\rho_0,-\theta_0)$$
$$- G_0(\rho,\theta;\rho_0,-\theta_0+\pi) + G_0(\rho,\theta;\rho_0,\theta_0-\pi), \tag{11}$$

$$G_3(\rho,\theta;\rho_0,\theta_0) = G_0(\rho,\theta;\rho_0,\theta_0) - G_0(\rho,\theta;\rho_0,-\theta_0)$$
$$- G_0(\rho,\theta;\rho_0,-\theta_0+2\pi/3) + G_0(\rho,\theta;\rho_0,\theta_0-2\pi/3)$$
$$+ G_0(\rho,\theta;\rho_0,\theta_0+2\pi/3) - G_0(\rho,\theta;\rho_0,-\theta_0-2\pi/3). \tag{12}$$

For general n, $n = 1, 2, 3, \ldots$, the Green's function G_n can be expressed as

$$G_n(\rho,\theta;\rho_0,\theta_0) = \sum_{k=0}^{n-1}[G_0(\rho,\theta;\rho_0,\theta_0+2k\pi/n) - G_0(\rho,\theta;\rho_0,-\theta_0-2k\pi/n)], \tag{13}$$

where it has been used that $G_0(\rho,\theta;\rho_0,\theta_0)$ is periodic in θ_0 with period 2π.

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